Random Variables

Definition. A **random variable** is a quantity whose value depends on chance

Definition. A **random variable** is a function from sample space to $\mathbb{R}$

Example

Suppose we have sample space $\Omega = \{a, b, c, d, e\}$ and $\mathbb{P}(a) = \mathbb{P}(b) = \mathbb{P}(c) = \mathbb{P}(d) = \mathbb{P}(e) = \frac{1}{5}$. Suppose we also have $X : \Omega \to \mathbb{R}$ such that $X(a) = 1, X(b) = 2, X(c) = 3, X(d) = 4, X(e) = 5$. X is a random variable.

Example

We toss a coin and roll a die and use their results as our sample space. What random variables could we have?

	H	T
1	(H,1)	(T,1)
2	(H,2)	(T,2)
3	(H,3)	(T,3)
4	(H,4)	(T,4)
5	(H,5)	(T,5)
6	(H,6)	(T,6)

Definition.

$$\mathbb{P}(X = x) = \mathbb{P}(\{\omega \in \Omega | X(\omega) = x\})$$

We say 'the probability X takes the value x'. It is standard to use capital letters to denote the random variable, and lower-case letters for the value it takes.

Example

$$\begin{aligned}\mathbb{P}(X = 3) &= \mathbb{P}(\{\omega \in \Omega | X(\omega) = 3\}) \\ &= \mathbb{P}(\{c\}) \\ &= \frac{1}{5}\end{aligned}$$

General Discrete Distribution

Definition. The **probability distribution** of a random variable is a mapping of possible values of the variable and the corresponding probabilities.

Since a random variable must take a value,

$$\sum_x \mathbb{P}(X = x) = 1 \text{ where the sum is over all possible values } X \text{ can take}$$

Example (A (not very) random variable)

$\mathbb{P}(X = 1) = 1$

Example (Discrete Uniform)

$\mathbb{P}(X = x) = \frac{1}{6}$ for $1 \leq x \leq 6$

Example

$\mathbb{P}(X = x) = cx(x+1)$ for $1 \leq x \leq 10$. What is c?

Example (Infinite discrete distribution)

$\mathbb{P}(X = x) = \dfrac{1}{2^x}, x = 1, 2, 3, \ldots$

Independence

Definition. We say random variables X and Y are **independent** iff

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

for all $x, y \in \mathbb{R}$

Tip

In other words, the **events** $\{\omega \in \Omega | X(\omega) = x\}$ and $\{\omega \in \Omega | Y(\omega) = y\}$ are independent

Example

X and Y are discrete random variables, with distributions

x	1	2	3
$P(X = x)$	0.2	0.5	0.3

y	0	1
$P(Y = y)$	0.4	0.6

(a) Find $P(X = 2 \text{ and } Y = 0)$.

(b) Find $P(X + Y = 3)$.

(a) Because X and Y are independent:

$$P(X = 2 \cap Y = 0) = P(X = 2) \times P(Y = 0) = 0.5 \times 0.4 = \mathbf{0.2}$$

(b) Combinations for $X + Y = 3$ *are:* $(X = 3, Y = 0)$ *and* $(X = 2, Y = 1)$.

- $P(X = 3, Y = 0) = 0.3 \times 0.4 = 0.12$
- $P(X = 2, Y = 1) = 0.5 \times 0.6 = 0.30$

Total Probability $= 0.12 + 0.30 = \mathbf{0.42}$

Example

The joint probability table for two discrete random variables S and T is shown below:

	$T = 1$	$T = 2$
$S = 1$	0.12	0.18
$S = 2$	0.28	0.42

(a) Find the marginal distributions of S and T (i.e., find $P(S = s)$ and $P(T = t)$ for all possible values).

(b) Determine whether S and T are independent. Justify your answer.

1. *Marginal Distributions:*
   - $P(S = 1) = 0.12 + 0.18 = \mathbf{0.3}$
   - $P(S = 2) = 0.28 + 0.42 = \mathbf{0.7}$
   - $P(T = 1) = 0.12 + 0.28 = \mathbf{0.4}$
   - $P(T = 2) = 0.18 + 0.42 = \mathbf{0.6}$
2. *Check independence condition $P(S = s)P(T = t) = P(S = s \cap T = t)$:*
   - $0.3 \times 0.4 = 0.12$ *(Matches $S = 1, T = 1$)*
   - $0.3 \times 0.6 = 0.18$ *(Matches $S = 1, T = 2$)*

   *The variables are **independent** because the condition holds for all cells.*

Expectation and Variance

Definition. The **expectation** (mean) of a random variable is

$$\mu = \mathbb{E}[X] = \sum_x x\mathbb{P}(X = x)$$

Example (Rolling a die)
We can model a roll of a fair die as a random variable X with $\mathbb{P}(X = i) = \frac{1}{6}$ for $i \in \{1, 2, 3, 4, 5, 6\}$. Then $\mathbb{E}[X] =$

Recall that for a set of data $\{x_i\}$, we had $\overline{x} = \dfrac{\sum f_i x_i}{\sum f_i}$, so we have

$$\begin{aligned}
\overline{x} &= \frac{\sum f_i x_i}{\sum f_i} \\
&= \frac{\frac{1}{n}\sum f_i x_i}{\frac{1}{n}\sum f_i} \\
&= \frac{\sum \frac{f_i}{n} x_i}{\sum \frac{f_i}{n}} \\
&\approx \frac{\sum \mathbb{P}(X = x_i) x_i}{\sum \mathbb{P}(X = x_i)} \\
&= \sum \mathbb{P}(X = x_i) x_i
\end{aligned}$$

Definition. The **variance** of a random variable X is defined to be:

$$\sigma^2 = \mathrm{Var}[X] = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$\mathrm{Var}[X] = \sum_x x^2 \mathbb{P}(X=x) - \left(\sum_x x\mathbb{P}(X=x)\right)^2$$

Example (Rolling a die)
We can model a roll of a fair die as a random variable X with $\mathbb{P}(X=i) = \frac{1}{6}$ for $i \in \{1,2,3,4,5,6\}$. Then $\mathrm{Var}[X] =$

Fact — For a general function $g : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[g(X)] = \sum_x g(x)\mathbb{P}(X=x)$$

Coding

Suppose X and Y are random variables, related to each other by $Y = aX + b$, where $a, b \in \mathbb{R}$ are constants. Then

Fact —

$$\mathbb{E}[Y] = \mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

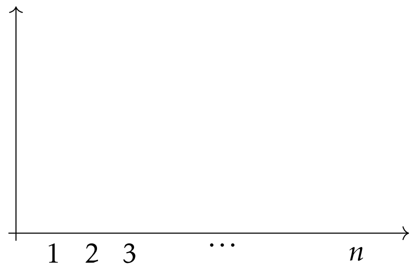
$$\text{Var}[Y] = \text{Var}[aX + b] = a^2\text{Var}[X]$$

Discrete Uniform Distribution

Definition. A random variable X is distributed with the **discrete uniform distribution** on $[n] = \{1, 2, \ldots, n\}$ if

$$P(X = i) = \begin{cases} \frac{1}{n} & \text{if } i \in \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

We often write $X \sim U(n)$, read X is *distributed* with the discrete uniform distribution



Fact — If $X \sim U(n)$

$$\mathbb{E}[X] = \frac{n+1}{2}$$
$$\text{Var}[X] =$$

Binomial Distribution

Example
Suppose we flip a fair coin 5 times, what is the probability we get 3 heads?

Definition. A random variable $X \sim B(n,p)$ is distributed with the **binomial distribution** with n trials and probability p of success if

$$P(X = i) = \begin{cases} \binom{n}{i} p^i (1-p)^{n-i} & \text{if } i \in \{0, 1, \dots, n\} \\ 0 & \text{otherwise} \end{cases}$$

There are two conditions to use a Binomial model:

1. Each trial is either a success or not
2. There is a *fixed* number of trials

We also need to make two assumptions:

1. Each trial has the same probability
2. Each trial is independent.

Example
On how many of the first 10 days of February will it rain?

Example
If I draw 13 cards from a deck of cards, how many red cards will there be?

Example

A fair, five-sided spinner is spun 20 times. What is the probability we get exactly the value 5 exactly twice?

Let X be the number of times we get the value 5.
We can model $X \sim B(20, \frac{1}{5})$ since each spin is independent, and is a success or failure with fixed probability. Then $\mathbb{P}(X = 2) = \binom{20}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^{18} = 0.1369\,(4\ d.p.)$

Mean and variance of Binomial distribution

Fact — If $X \sim B(n, p)$, and $q = 1 - p$

$$\mu = \mathbb{E}[X] = np$$
$$\sigma^2 = \mathrm{Var}[X] = npq$$

Linearity of mean and variance

Theorem

If X and Y are random variables:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Theorem

If X and Y are independent random variables (ie $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$:

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$$

Linearity of mean and variance (Attempt 2)

Definition. We say $X \sim \text{Bernoulli}(p)$ or $X \sim B(1,p)$ is a **Bernoulli random variable**

> Fact — If $X \sim B(1,p)$, and $q = 1-p$ then $\mathbb{E}[X] = p, \text{Var}[X] = pq$

> Fact — If $X \sim B(n,p)$, and $q = 1-p$
>
> $$\mu = \mathbb{E}[X] = np$$
> $$\sigma^2 = \text{Var}[X] = npq$$

Geometric Distribution

Example

Suppose we are flipping a fair coin until it comes up heads. What is the probability we need k flips?

Definition. A random variable $X \sim \text{Geo}(p)$ is distributed with the **geometric distribution** with probability p of success if

$$P(X = k) = \begin{cases} (1-p)^{k-1}p & \text{if } k \in \{1,2,3,\ldots\} \\ 0 & \text{otherwise} \end{cases}$$

Example

A darts player must start a game by hitting a certain section of the dart-board, The probability that any single dart hits this section is $\frac{2}{5}$. The player keeps throwing darts until he hits the required section. Calculate the probability that he needs:

(a) exactly 5 throws,

(b) at most 2 throws,

(c) at least 8 throws,

(d) at most 10 throws.

(e) How many throws does the player need to make to be at least 99.999% sure of hitting the required section?

(f) What assumptions have you made in using the geometric distribution to model this situation?

Fact — If $X \sim \text{Geo}(p)$

$$\mu = \mathbb{E}[X] =$$
$$\sigma^2 = \text{Var}[X] =$$

Example

How many throws will the darts player make in expectation?

$\mathbb{E}(X) = \frac{1}{p} = \frac{5}{2} = 2.5$

Negative Binomial Distribution

Example

Suppose we are flipping a fair coin, how many flips do we need in order to have flipped 2 heads?

Definition. A random variable $X \sim \text{NB}(r, p)$ is distributed with the **negative binomial distribution** which gives the probability of k failures until r successes where the probability of success is p

$$P(X = k) =$$

Example

If $X \sim \text{NB}(r, p)$, calculate $\mathbb{E}(X)$ and $\text{Var}(X)$

Hypergeometric Distribution

Example

Suppose we draw 3 cards from a standard deck of 52. What is the probability we get (exactly) 2 aces?

Definition. A random variable $X \sim \text{Hypergeometric}(N, K, n)$ is distributed with the **hypergeometric distribution** which gives the probability of k successes from n draws, *without replacement*, from a population of size N, with K possible successes

$$P(X = k) =$$

Example

If $X \sim \text{Hypergeometric}(N, K, n)$, calculate $\mathbb{E}(X)$ and $\text{Var}(X)$